Annalen
der Physik

Acoustic and Elastic Waves in Random Media - CPA

M. Kafesaki[1] **and E.N. Economou**[1,2]

[1]Institute of Electronic Structure and Laser, Foundation for Research and Technology Hellas, P.O. Box 1527, 71110, Heraklion, Crete, Greece.
[2]Physics Department, University of Crete, Greece.

Received 6 October 1998, revised version, accepted.

Abstract. We study the propagation of acoustic and elastic waves in random composites consisting of spherical inclusions in a homogeneous elastic host (fluid or solid), using various extensions of the Coherent Potential Approximation (CPA) method, well-known from the electronic problem. We calculate the phase velocity, the scattering mean free path and the transport velocity. The results are compared with experimental data and accurate computational results.

Keywords: Waves, Acoustics, Elasticity, Random, CPA.

The Coherent Potential Approximation (CPA) method, well known from the study of the electron wave propagation in disordered solids, has been extended in recent years to the problem of classical wave (CW) propagation in random composites. The attempts to extend the CPA in the CW case gave rise to various versions of the method which have been applied mainly to simple scalar (SS) and to electromagnetic (EM) waves, with considerable success.

On the other hand, acoustic[1] (AC) and elastic (EL) waves have received less attention. The limited experimental data in connection with computational difficulties (arising from the mixed longitudinal and transverse vector character of the elastic waves and the resulting scattering induced mode conversion plus the two or three parameters characterizing an acoustic or elastic medium) seem to be responsible for this inattention to the field.

Recently, various versions of the CPA were applied to AC wave propagation in a random composite consisting of glass spheres in water [1, 2]. They gave results in very good agreement with recent experimental data [2]. Motivated by this success and by some older experimental data, we apply in the present work the same CPA versions as in ref. [1] to the case of AC and mainly EL wave propagation. Our study concerns binary composites consisting of spheres of radius a (scattering material), randomly placed in a host material. (In the following discussion the scattering material will be characterized by the subscript i (=in) and the host by (o (=out).)

Methods: The main idea of the CPA is the replacement of the random medium by a homogeneous one (effective medium) characterized (in the simplest case of scalar CW)

[1]With *acoustic* we mean density waves in macroscopically fluid systems.

by a complex propagation vector $\mathbf{q}_e$ (the subscript e denotes the effective medium). $q_e\,(=|\mathbf{q}_e|)$ is calculated self-consistently, by requiring the vanishing of the average forward scattering amplitude, $f(0)=f(\theta=0)$, of a plane wave, scattered by various scattering units of the actual material (scattering configuration), embedded in the effective medium.

The different ways of choosing these scattering units give rise to various versions of the method, e.g. the simple CPA (S-CPA) and the coated CPA (C-CPA) [**3**] which are used in the present work. Within the S-CPA two types of scattering units are considered. The first one is an actual scattering material sphere, which is embedded in the effective medium with probability $p_1 = f_s$ (f_s: volume fraction of the spheres in the actual composite). The second is a sphere of the same size formed by the host material, embedded in the effective medium with probability $p_2 = 1 - f_s$. For the C-CPA the scattering units are a scattering material sphere coated by a host coating (coated sphere of external radius r_1) and a simple host material sphere (of radius r_2) with corresponding probabilities p_1 and p_2. This choice represents the fact that in the actual medium the two materials (scattering and host) are not topologically equivalent. For the radii r_1 and r_2 and the probabilities p_1, p_2, see ref. [**3**].

Taking into account the above, the CPA condition for the determination of the q_e assumes the form

$$< f(0) >_c = p_1 f_1(0) + p_2 f_2(0) = 0 \tag{1}$$

with $f_1(0), f_2(0)$ the forward scattering amplitudes when a plane wave is scattered by the two scattering units.

From q_e, one can calculate immediately the phase velocity, $c_{ph} = \omega/\Re(q_e)$, and the scattering mean free path , $l_s = 1/[2\Im(q_e)]$. Our aim here is to calculate these quantities for acoustic and mainly for elastic wave propagation, testing thus both S-CPA and C-CPA in these more complicated cases.

Particular features of the AC and EL waves: In the EL wave case a homogeneous medium is characterized by two propagation vectors (two velocities), one for longitudinal waves, $q_{le} = \omega/c_{le}$, and one for transverse, $q_{te} = \omega/c_{te}$. Thus, for the determination of the elastic effective medium one has to calculate q_{le}, q_{te} plus the density of the effective medium, ρ_e, which is essential for the calculation of q_{le}, q_{te}. This requires at least two more conditions. One additional condition is obtained if one takes into account that the EL wave forward scattering amplitude is different for a longitudinal than for a transverse wave. Single scattering study shows that there is no mode mixing in the forward direction. Applying Eq.(1) to $f_{ll}(0)$ and $f_{tt}(0)$[2] (the first subscript denotes the type of the incident wave and the second the type of the scattered), one obtains two complex equations, enough for the calculation of q_{le}, q_{te}. The density of the effective medium, ρ_e, can be calculated either self consistently or can be approximated by either the average density, $f_s\rho_i + (1-f_s)\rho_o$, or by the long wavelength limit results. Our attempts for a self consistent calculation of the ρ_e didn't have many successes. We tried this by requiring the vanishing of the converted scattering amplitude, f_{lt} in a direction other than the forward ($f_{lt} \propto f_{tl}$; this vanishes in the forward direction). In most of the cases convergence difficulties appeared. In the cases in which a result could be obtained, this was very close to what is obtained by using the approximate expressions for the density. The same problem exists also in the pure longitudinal AC wave case. For AC waves we use Eq.(1) to calculate the

[2] For elastic waves the scalar amplitude f_{tt} is given by $\hat{\mathbf{n}}\mathbf{f}_{tt}$, where $\hat{\mathbf{n}}$ is the polarization direction of the incident wave and $\mathbf{f}_{tt}$ the vector transverse scattering amplitude.

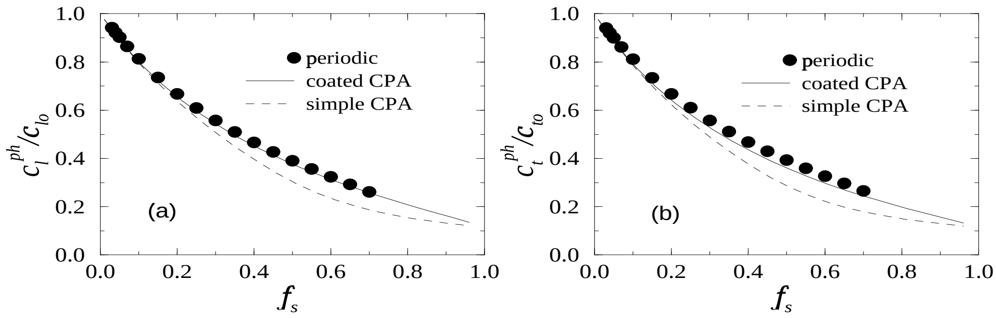

Fig. 1: (a) Longitudinal wave phase velocity, c_l^{ph}, for a random system of spheres in an elastic medium vs volume fraction of the spheres, f_s. Long wavelength limit results. For the system: $\rho_o/\rho_i = 1/4$, $c_{lo}/c_{li} = 8.66$, $c_{li(o)}/c_{ti(o)} = 1.415$. (b) Transverse wave phase velocity, c_t^{ph}, for the same system as in (a).

$q_e = q_{le}$ while for the determination of the ρ_e we use the corresponding to the elastic case approximations.

Results: We compare, first, results based on the C-CPA and S-CPA in the EL wave case as shown in Fig.1. Fig.1(a) shows the phase velocity as a function of the volume fraction of the scatterers for longitudinal waves propagating in a random system of solid spheres in a solid host. Fig.1(b) presents the similar results for transverse waves. These results have been obtained in the long wavelength limit. The S-CPA (dashed line) and the C-CPA (solid line) curves are compared with accurate results (circles) concerning the fcc periodic system formed by the same material combination. Due to the long wavelength, λ ($\lambda \gg$ the characteristic lengths of the system), the wave can not distinguish a periodic from a random system. Thus, the CPA results ought to coincide with the accurate results of the periodic system. As one can see in Fig.1, this is the case for the coated CPA. Moreover, S-CPA, while it doesn't fail, seems to be less successful.

In Fig.2 we show the phase velocities for longitudinal and transverse waves, propagating in a random system consisting of Pb spheres (of volume fraction 5%) in epoxy host (for Pb (epoxy): $\rho = 11.3\,(1.202)$ g/cm^3, $c_l = 2.21\,(2.64)$ km/s, $c_t = 0.86\,(1.2)$ km/s). Here, both S-CPA results (dashed line) and C-CPA results (solid line) are in very good agreement with the corresponding experimental data (triangles in Fig.2(a)) [4]. The dips in the phase velocities at $k_{lo}a \approx 0.3$ ($k_{lo} = \omega/c_{lo}$) are close to the first resonance in the Pb in epoxy single scattering cross section [5]. Moreover, the CPA results exhibit some small peaks which are not present in the experimental data. These peaks are also due to the single scattering resonances which apparently are wiped out by multiple scattering effects.

Going to higher concentrations CPAs become less successful. As the concentration increases the enhanced multiple scattering effects become more pronounced.

Limiting case results: Applying Eq.(1) in the low concentration and the long wavelength limit, we retrieve well known analytical results for these two limiting cases.

Low concentration limit: In low concentrations of the scatterers we can take the effective medium parameters to be equal to the host medium parameters plus small correction terms. Expanding Eq.(1) in these terms (for the full elastic case) the following relations result, connecting the scattering mean free paths, $l_{l(t)} = 1/[2\Im(q_{l(t)e})]$,

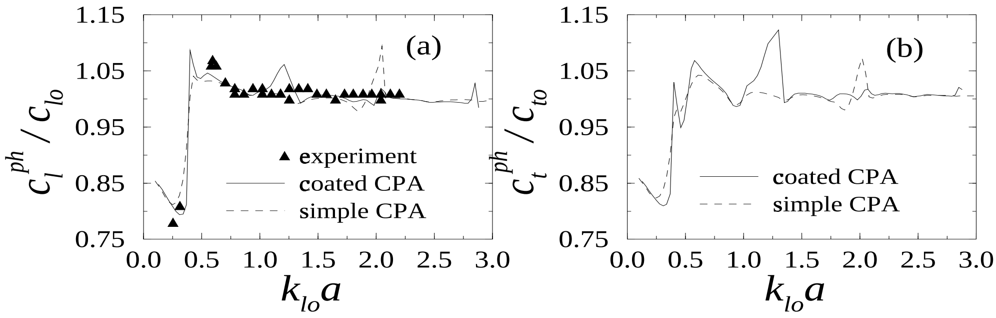

Fig. 2: Longitudinal (a) and transverse (b) phase velocity as a function of $k_{lo}a = \omega a/c_{lo}$ for Pb spheres in epoxy. a spheres radius, c_{lo} longitudinal velocity in Pb.

with the single sphere scattering cross sections:

$$l_l = 1/n_s\sigma_l, \qquad l_t = 1/n_s\sigma_t. \tag{2}$$

n_s is the number density of the scatterers and σ_l, σ_t the single scattering cross section, supposing longitudinal and transverse incident wave, respectively. Relations (2) are quite analogous to the SS and EM wave case. The new feature is that, in low concentrations, the $l_l(l_t)$ is related only with $\sigma_l(\sigma_t)$.

Long wavelength limit: Analyzing a plane wave in spherical waves, the scattering amplitudes f_{ll}, f_{tt} can be written as a sum of partial scattering amplitudes, f_{nll}, f_{ntt} each one from the scattering of each partial spherical wave (n). In the long wavelength limit, f_{ll} and $f_{tt} \to 0$ as ω^2. The ω^2 contribution comes from the $n=0$ and $n=1$ partial waves for the f_{ll}, and from the $n=1$ for the f_{tt}. Applying the CPA condition (Eq.(1)), in the long wavelength limit, for the $n=0$ and $n=1$ partial forward scattering amplitudes and for the configuration of the simple CPA, we obtain:

(a) For a random system of *solids*:

$$\rho_e = f_s\rho_i + (1-f_s)\rho_o\,, \qquad f_s\frac{B_i - B_e}{3B_i + 4\mu_e} + (1-f_s)\frac{B_o - B_e}{3B_o + 4\mu_e} = 0. \tag{3}$$

B is the bulk modulus and μ the shear modulus.

(b) For a random system of *fluids*:

$$f_s\frac{\rho_i - \rho_e}{2\rho_i + \rho_e} + (1-f_s)\frac{\rho_o - \rho_e}{2\rho_o + \rho_e} = 0\,, \qquad \frac{1}{B_e} = \frac{f_s}{B_i} + \frac{1-f_s}{B_o}. \tag{4}$$

The second equation in (4) is the well known Wood's low.

Energy velocity - Energy CPA (E-CPA): In the case of enhanced multiple scattering the coherent part of the wave becomes negligible. Thus, the phase velocity - velocity related with coherent propagation - has no meaning. In that case the propagation is described by the energy transport velocity, v_E, defined by $D = v_E l^*/3$ (D: diffusion coefficient, l^* transport mean free path). Experimental results in SS and EM wave propagation have shown that v_E, as a function of frequency, exhibits pronounced dips close to the single scattering resonances in low concentrations while as the concentration increases the dependence of the v_E on the frequency becomes smoother [6]. The dips in the low concentrations were attributed to the delay of the wave inside the scatterers while as the concentration increases the scatterers provide an additional path for the propagation permitting the wave to hop from one to the neighboring scatterer, and reducing thus the above mentioned delay.

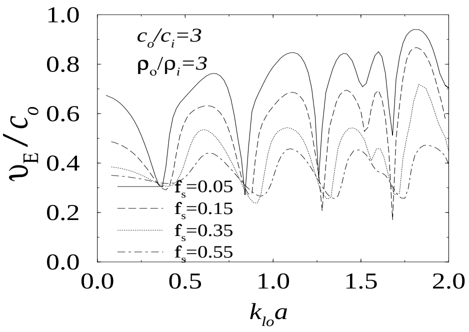

Fig. 3: Energy velocity vs $k_{lo}a = \omega a/c_{lo}$ for a random system of fluid spheres in a fluid. f_s: sphere volume fraction.

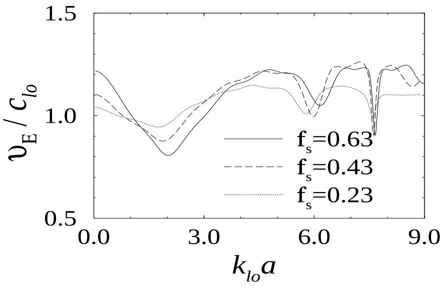

Fig. 4: Energy velocity vs $k_{lo}a = \omega a/c_{lo}$ for glass spheres in water. f_s: volume fraction of the spheres.

The velocity v_E has been calculated accurately for scalar and EM waves *only* in the low concentration limit [6]. One of the attempts to extend its calculation in higher concentrations led to the energy CPA (E-CPA) [7] which we discuss next. The main idea of the E-CPA is that in a random system energy should be homogeneous in scales larger than the typical lengths of the system. Thus, energy CPA calculates first a homogeneous medium with energy density equal to the average energy density of the random medium. After that, the v_E is calculated by considering the scattering relative to this homogeneous medium [7, 1].

The attempts to apply the energy CPA in the EL wave case were completely unsuccessful. This was due to the full vector character of the EL waves with the two different velocities and the mode conversion effect. Calculation of the v_E through the E-CPA has as a result two different energy velocities one related with longitudinal wave propagation and the other with transverse. The existence of two different energy velocities for EL waves in the strong scattering (diffusive) regime has no physical meaning as, in this regime, successive conversions of the longitudinal to transverse wave and vice versa take place. The question of the proper energy velocity for EL wave propagation (or maybe which combination of our two energy velocities can give a proper v_E) is an open problem.

Below, we present v_E results for AC wave propagation (for the calculation of v_E through E-CPA for AC waves see [1]). Our main aim is to check if the dependence of v_E on the volume fraction is similar to what has been found for the EM and scalar wave propagation.

Fig.3 shows the v_E as a function of the frequency for a random system consisting of fluid spheres in a fluid host, with $c_o/c_i = 3$, $\rho_o/\rho_i = 3$, in four different concentrations. As one can see, the picture is quite analogous to what happens in the SS and EM wave propagation: dips in low concentrations (close to the single scattering resonances) and smoothness of the v_E as the concentration increases. As additional results show, these features describe the v_E dependence on the volume fraction in the cases where the corresponding single scattering cross section exhibits strong resonances associated with high concentration of the wave energy inside the sphere. The explanation is the same to what is mentioned above for the EM waves.

A different picture is that of Fig.4 where we show v_E results for a system consisting of glass spheres in water. The different to the above and unexpected dependence on the volume fraction can be attributed here to the fact that, in this system, the wave does penetrate considerably into the glass sphere at the resonances, as the single scattering study has shown [1].

In conclusion: The CPA describes the experimental results reasonably well. By considering the single scattering cross section one obtains simple physical explanations even for the minor discrepancies between the CPA results and the experimental data.

References

[1] M. Kafesaki and E. N. Economou, Europhys. Lett. **37** (1997) 7.

[2] J. H. Page, P. Sheng, H. P. Schriemer, I. Jones, X. Jing and D. A. Weitz, Science **271** (1996) 634.

[3] C. M. Soukoulis, S. Datta, E. N. Economou, Phys. Rev. B **49** (1994) 3800.

[4] V. K. Kinra, E. L. Ker and S. K. Datta, Mech. Res. Commun **9** (1982) 109.

[5] M. Kafesaki, M. M. Sigalas and E. N. Economou, Sol. St. Commun. **96** (1995) 285.

[6] M. P. van Albada, B. A. van Tiggelen, Ad Lagendijk, A. Tip, Phys. Rev. Lett. **66** (1991) 3132; B. A. van Tiggelen and Ad Lagendijk, Europhys. Lett. **23** (1993) 311.

[7] K. Bush, C. M. Soukoulis, Phys. Rev. Lett. **75** (1995) 3442.