Effective Medium Theory of Left-Handed Materials

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We analyze the transmission and reflection data obtained through transfer matrix calculations on metamaterials of finite lengths, to determine their effective permittivity ϵ and permeability μ. Our study concerns metamaterial structures composed of periodic arrangements of wires, cut wires, split ring resonators (SRRs), closed SRRs, and both wires and SRRs. We find that the SRRs have a strong electric response, equivalent to that of cut wires, which dominates the behavior of left-handed materials (LHM). Analytical expressions for the effective parameters of the different structures are given, which can be used to explain the transmission characteristics of LHMs. Of particular relevance is the criterion introduced by our studies to identify if an experimental transmission peak is left or right handed.

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Recently, there have been many studies about metamaterials that have a negative refractive index n. These materials, called left-handed materials (LHMs), theoretically discussed by Veselago [1], have simultaneously negative electrical permittivity ϵ and magnetic permeability μ. Such materials consisting of split ring resonators (SRRs) and continuous wires were first introduced by Pendry [2,3], who suggested that they can also act as perfect lenses [4].

Since the original microwave experiment by Smith *et al.* [5], which first materialized Pendry's proposal, various new samples were prepared [6,7] (composed of SRRs and wires) all of which have been shown to exhibit a passband in which it was assumed that ϵ and μ are both negative. This assumption was based on measuring independently the transmission, T, of the wires alone, and then the T of the SRRs alone. If the peak in the combined metamaterial composed of SRRs and wires were in the stop bands for the wires alone (which corresponds to negative ϵ) *and* for the SRRs alone (which is thought to correspond to negative μ) the peak was considered to be left-handed (LH). Further support to this interpretation was provided by the demonstration that some of these materials exhibit negative refraction of electromagnetic waves [8]. Subsequent experiments [9] have reaffirmed the property of negative refraction, giving strong support to the interpretation that these metamaterials can be correctly described by negative permittivity and negative permeability. However, as we show in the present study, this is not always the case. The combined system of wires and SRRs exhibits synergy of the two components as a result of which its effective plasma frequency, ω'_p, is much lower than the plasma frequency of the wires, ω_p.

There is also a significant amount of numerical work [10–13] in which the complex transmission and reflection amplitudes are calculated for a finite length of metamaterial. Using these data a retrieval procedure can then be applied to obtain the effective permittivity ϵ and permeability μ, under the assumption that the metamaterial can be treated as homogeneous. This procedure confirmed [14,15] that a medium composed of SRRs and wires could indeed be characterized by effective ϵ and μ whose real parts were both negative over a finite frequency band, as was the real part of the refractive index n.

In this Letter, we study the transmission of periodic systems made up of wires alone, of SRRs alone, and of combined systems of wires and SRRs (LHMs). Through a very detailed retrieval scheme [14], the effective ϵ and μ of those systems are obtained. It is shown that the SRRs have also an electric response, in addition to their magnetic response, which was first described by Pendry [3]. The electric response of the SRRs is demonstrated by closing their air gaps, and therefore destroying their magnetic response. In fact, it is shown that the electric response of the SRRs is identical to that of cut wires. Analytical expressions for the effective ϵ of wires and SRRs as well as for the effective μ of SRRs are given. Using these analytical expressions one is able to reproduce the low frequency transmission, T, and reflection, R, characteristics of LHMs. Even the minor details in T and R observed in the simulations can be analytically explained. The main power of the present analysis though

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is that it gives an easy criterion to identify if an experimental transmission peak is LH or right handed (RH): If the closing of the gaps of the SRRs in a given LHM structure removes the peak close to the position of the SRR dip from the T spectrum, this implies that the T peak is indeed left handed. This criterion is very valuable in experimental studies, where one cannot easily obtain the effective ϵ and μ and holds provided the magnetic resonance frequency ω_m is well separated from ω_p'. Our criterion is used experimentally and is found that some T peaks that were thought to be LH, turn out to be right handed [16].

We use the transfer matrix method to simulate numerically the transmission properties of the metamaterials. A representative result is shown in Fig. 1, presenting transmission spectra for a usual SRR and wire metamaterial and for its isolated constituents. As expected, the SRRs system [Fig. 1(a)] shows a well defined T dip near $\omega a/c = 0.04$ (a is the discretization length, related to the unit cell as explained in the caption of Fig. 1, and c is the vacuum speed of light), due to its magnetic resonance, which gives a negative effective μ in the dip regime, while its effective ϵ is positive in this frequency range. This interpretation has been verified with the retrieval procedure. The lattice of continuous wires alone shows no transmission up to the plasma frequency $\omega_p a/c = 0.09$ (see the dotted line in Fig. 1(b)] due to a negative effective ϵ. Combining the SRRs and the wires systems we find what is shown in Fig. 1(b)—solid line: no transmission at low frequencies (due to negative ϵ), a LH transmission peak (where both μ and ϵ are negative) that roughly corresponds to the transmission dip of the isolated SRRs, followed by a second transmission gap (where $\mu \geq 0$, $\epsilon \leq 0$) and terminated by a right-handed transmission shoulder (at about 0.06). The position of the last right-handed shoulder, which does not coincide with the ω_p of the isolated wires, shows that the SRRs contribute also to the electric response of the combined system, something that has never been considered previously. This SRR contribution leads to a new plasma frequency ω_p', smaller than ω_p. The present interpretation of the data is supported again by results obtained from the retrieval procedure. Thus, combining the resonant behavior in $\mu(\omega)$ of the SRRs with the negative behavior of $\epsilon(\omega)$ due to the wires only is clearly inadequate; the resonant contribution to $\epsilon(\omega)$ due to the SRRs has to be added.

Furthermore, the interaction of the electric response of wires and SRRs also has to be taken into account. Some indication that there exists such a non-negligible electric interaction between the SRRs and the wires comes from the comparison of the T for the in-plane system (i.e., wires next to the SRRs, as in Fig. 1(b)], with the T for an off-plane system (wires behind SRRs; see Fig. 1(c), solid line). It is found that for the in-plane case the first, left-handed transmission peak is much narrower and that the ω_p' is higher than for the off-plane case. The reduction

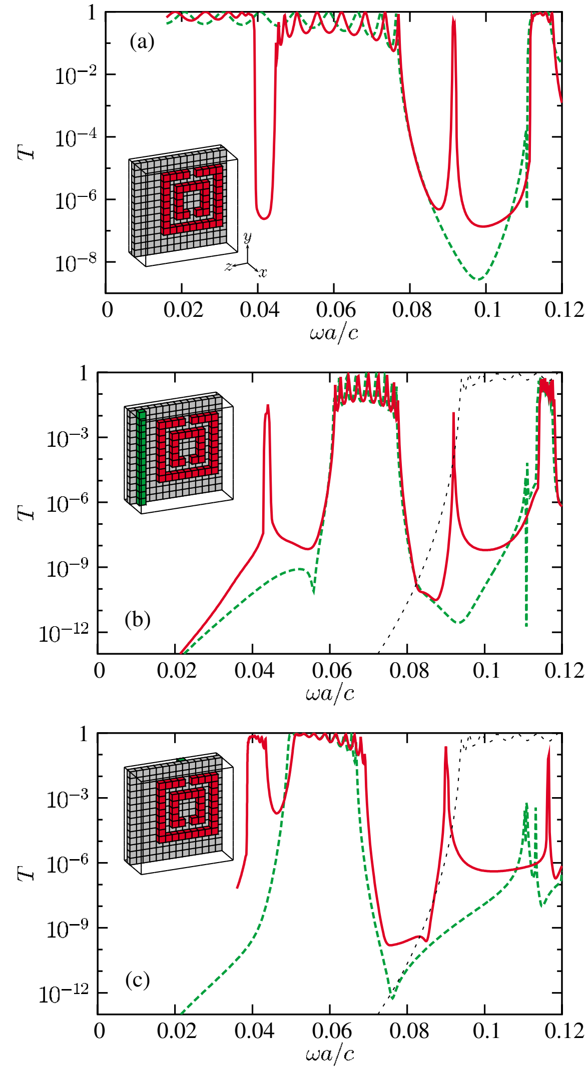

FIG. 1 (color online). (a) Frequency dependence of the transmission coefficient of a lattice of SRRs (solid line) and of closed SRRs (dashed line). (b) As in (a) for a metamaterial of SRRs plus wires (solid line) and of closed SRRs plus wires (dashed line), with the wires next to the SRRs. The dotted line shows the transmission for wires only. (c) As in (b) but with the wires in the opposite face of the SRRs. The insets show the geometry of the unit cell. The relative permittivity for the metals is taken to be $\epsilon_m = (-3 + i5.88)10^5$ and for the dielectric board $\epsilon_b = 12.3$. All relative permeabilities are one. The unit cell size is $6a \times 14a \times 14a$. a is the discretization length and c the light velocity in air.

of ω_0 [the electric resonance frequency of the SRRs; see Eq. (3) below] and consequently of the ω_p' [Fig. 1(c)] suggests that there is a significant coupling between the electric response of the wires and that of the SRRs in the off-plane case.

Since all the metals, the vacuum, and the dielectric boards are nonmagnetic materials, the only source of the major magnetic response of the LHM are the circulating induced currents in the metallic rings of the SRRs, which are forced to oscillate due to the gaps acting like capacitors. Therefore, a simple way to study the combined electric response of the LHM is to close the gaps in the rings of the SRRs. This destroys their magnetic resonance without substantially affecting their electric response. Closing the gaps, i.e., eliminating the capacitors from the rings of the SRRs, the induced circulating currents inside the SRRs are still allowed to flow but cannot oscillate independently of the external electromagnetic (EM) field anymore. The changes in the electric response from the closing of the SRRs are expected to be weak [indeed, ω_0 moves at most by 3% as can be seen in Fig. 1(c)], since only a very small amount of metal is added (in the gaps of the SRR). We have checked all these ideas by calculating the transmission with closed SRRs. The results (see the dashed lines in Fig. 1) are almost the same as the ones for the open SRRs. The only important difference is the disappearance of the first dip in Fig. 1(a), and the first peak in Figs. 1(b) and 1(c), as well as of the peak at about 0.09. The features that disappeared are due to the magnetic response of the SRRs. This was also confirmed with our retrieval procedure, which clearly showed that, indeed, the dip in Fig. 1(a) and the peaks in Figs. 1(b) and 1(c) are due to the magnetic response of the SRRs. The most important point of this analysis is that the combination of the wires with the SRRs can lower, in principle, the ω_p of the isolated wires to values very close to the frequency of the magnetic response, ω_m, of the SRRs. In some cases it is possible that ω'_p is lower than the ω_m, and transmission peaks appear at low frequencies which are not LH but RH. This also has been confirmed experimentally [16].

Through our detailed transmission studies and the retrieval procedure for obtaining the effective ϵ and μ of the different structures, we were able to obtain analytic expressions for the effective ϵ and μ for all the structures. In particular, as expected from effective medium arguments, an array of metallic wires exhibits the frequency-dependent plasmonic form

$$\epsilon_{\text{eff}}^{\text{wire}}(\omega) = 1 - \frac{\omega_p^2}{\omega^2 + i\omega\gamma}. \quad (1)$$

For the frequency dependence of the magnetic response of the SRR we obtain in the case of a single resonance [17] that

$$\mu_{\text{eff}}^{\text{SRR}}(\omega) = 1 - \frac{\omega_m'^2 - \omega_m^2}{\omega^2 - \omega_m^2 + i\omega\gamma}. \quad (2)$$

We have argued above and we have shown by introducing cut wires instead of SRRs that the electric response of the SRRs is equivalent to that of cut wires. Therefore, for the frequency dependence of the electric response of the SRR we obtain that [17]

$$\epsilon_{\text{eff}}^{\text{SRR}}(\omega) = 1 - \frac{\omega_p^2 - \omega_0^2}{\omega^2 - \omega_0^2 + i\omega\gamma}. \quad (3)$$

The magnetic resonance of the SRR is due to the oscillation of circular currents inside the metallic rings, and is determined by the inductance of the loop (enclosed area) and the capacitance of the gaps (mainly the gap width) in the rings. The second major contribution of the SRR, its electric cut-wire resonance, is due to the oscillation of linear currents along the sides of the SRR which are parallel to the external electric field. It is basically determined by the size of the SRR in the direction of the electric field. In comparison with the response of the continuous wire, Eq. (1), for a cut wire the electric resonance is shifted from zero to some $\omega_0 \neq 0$ because of the additional depolarization field due to its finite extent.

The electric response of the LHM is to first approximation equal to the sum of the electric response of the wires [given by Eq. (1)] and the electric cut-wire response of the SRRs [given by Eq. (3)], which may be approximated by the closed SRRs. The electric interaction between wires and SRRs has to be taken into account in order to be more realistic. Since the wires or the cut wires do not have any magnetic response, the magnetic response of the LHM is just the SRR magnetic resonance, given by Eq. (2). In Fig. 2 we present in a compact schematic form the response of the various "components" of a metamaterial of SRRs and wires to an EM field. This response is shown through the frequency dependence of the transmission coefficient (left panels), dielectric function (middle panels), and magnetic permeability (right panels). The first row shows the response of a periodic system of infinite wires. This response is analogous to that of a bulk metal; i.e., there is a cutoff frequency ω_p above which $\epsilon(\omega)$ becomes positive from negative, and thus the system becomes "transparent" to the EM radiation. The system does not have any magnetic response. The second row shows the response of a system of cut wires (wires much shorter than the wavelength of the EM wave), or closed SRRs. The difference with the continuous wires is that here the negative ϵ regime has also a lower edge $\omega_0 \neq 0$, due to the finite nature of the wires [15]. The $\epsilon(\omega)$ has the form shown in the middle graph. Again no magnetic response. The third row shows the response of a periodic system of (single-ring) SRRs. Their electric response is cut-wire-like (as in the second row), and their magnetic response has a resonance at ω_m, where the magnetic permeability (μ) jumps from positive to negative values. The transmission (left panel) becomes finite in the regions of positive product $\epsilon\mu$ and goes to zero for negative $\epsilon\mu$. The relative order of the characteristic frequencies (ω_m, ω_p, ω_0) depends on the parameters of each specific system. The fourth row shows the response of a system of infinite wires plus cut wires (closed SRR). This system contains, in fact, all the electric response of the LHM, without its magnetic response. The combined electric response of wires and cut wires leads to a new cutoff frequency, ω'_p, much lower than ω_p. The last

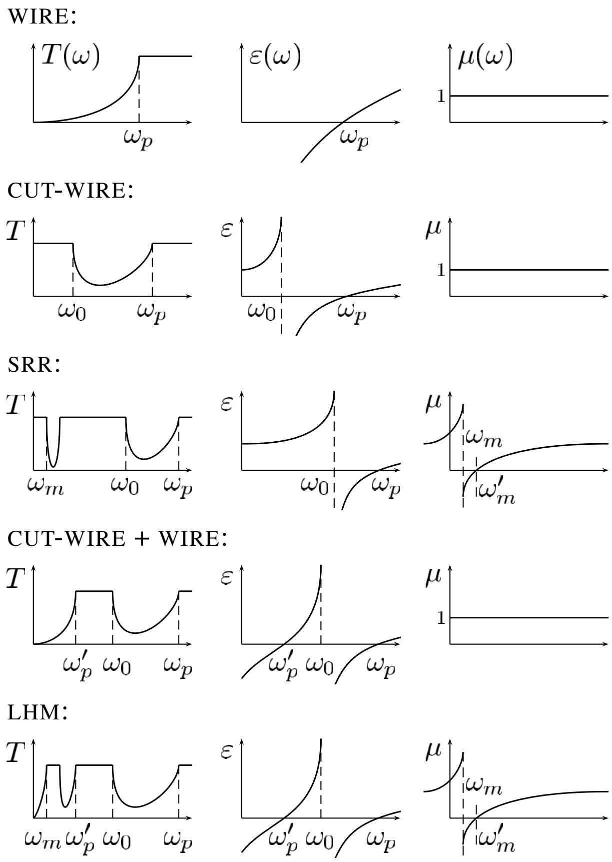

FIG. 2. Each row shows the response of each one of five periodic systems to electromagnetic wave. This response is shown through the frequency dependence of transmission coefficient (left panels), dielectric function (middle panels), and magnetic permeability (right panels).

row shows the full response of the LHM; its electric response is that of a system of wires plus cut wires (fourth row) and its magnetic response is that of a periodic system of SRRs (third row). Simulations (see Fig. 1) and the retrieval procedure have verified all aspects shown schematically in Fig. 2 (for the explanation of the peak at 0.09 see Ref. [17]).

We have systematically studied the transmission properties of LHMs composed of SRRs and continuous wires, and also of their various components separately. A retrieval procedure was used to obtain the effective parameters for each case. We found that the electric response of a LHM is the sum of electric responses of the wires and the SRRs. This changes the existing picture where the electric response of a LHM was attributed only to the wires, and provides a valid scheme for interpreting experimentally observed transmission peaks.

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