Vector fields

Light is a 3D vector field.

A 3D vector field \( \vec{F} \) assigns a 3D vector (i.e., an arrow having both direction and length) to each point in 3D space.

A light wave has both electric and magnetic 3D vector fields:

Maxwell’s Equations and Light Waves

Longitudinal vs. transverse waves

Div, grad, curl, etc., and the 3D Wave equation

Derivation of the wave equation from Maxwell’s Equations

Why light waves are transverse waves

Why we neglect the magnetic field

Irradiance, superposition and interference

Longitudinal vs. Transverse waves

Longitudinal:
Motion is along the direction of propagation

Transverse:
Motion is transverse to the direction of propagation

Space has 3 dimensions, of which 2 directions are transverse to the propagation direction, so there are 2 transverse waves in addition to the potential longitudinal one.

The 3D vector wave equation for the electric field

\[
\nabla^2 \vec{E} - \mu \varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0
\]

Note the vector symbol over the \( E \).

This is really just three independent wave equations, one each for the \( x \), \( y \), and \( z \) components of \( E \).

which has the vector field solution:

\[
\vec{E}(\vec{r}, t) = \vec{A} \exp \left[ i \left( \vec{k} \cdot \vec{r} - \omega t - \phi \right) \right]
\]

\[
\vec{E}(\vec{r}, t) = \vec{E}_0 \exp \left[ i \left( \vec{k} \cdot \vec{r} - \omega t \right) \right]
\]
**Waves using complex vector amplitudes**

We must now allow the complex field \( \vec{E} \) and its amplitude \( E_0 \) to be vectors:

\[
\vec{E}(r,t) = E_0 \exp \left[ i (k \cdot r - \omega t) \right]
\]

Note the arrows over the \( E's! \)

The complex vector amplitude has six numbers that must be specified to completely determine it!

\[
\vec{E}_0 = (\text{Re}\{E_x\} + i \text{Im}\{E_x\}, \text{Re}\{E_y\} + i \text{Im}\{E_y\}, \text{Re}\{E_z\} + i \text{Im}\{E_z\})
\]

**Div, Grad, Curl, and all that**

Types of 3D vector derivatives:

The Del operator:

\[
\nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)
\]

The Gradient of a scalar function \( f \):

\[
\nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)
\]

The gradient points in the direction of steepest ascent.

The Laplacian of a scalar function:

\[
\nabla^2 f = \vec{\nabla} \cdot \vec{\nabla} f = \vec{\nabla} \cdot \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}
\]

The Laplacian of a vector function is the same, but for each component of \( f \):

\[
\nabla^2 \vec{f} = \left( \frac{\partial^2 f_x}{\partial x^2} + \frac{\partial^2 f_x}{\partial y^2} + \frac{\partial^2 f_x}{\partial z^2}, \frac{\partial^2 f_y}{\partial x^2} + \frac{\partial^2 f_y}{\partial y^2} + \frac{\partial^2 f_y}{\partial z^2}, \frac{\partial^2 f_z}{\partial x^2} + \frac{\partial^2 f_z}{\partial y^2} + \frac{\partial^2 f_z}{\partial z^2} \right)
\]

The Laplacian tells us the curvature of a vector function.
The equations of optics are Maxwell’s equations.

\[ \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon} \\
\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \\
\nabla \cdot \vec{B} = 0 \\
\nabla \times \vec{B} = \mu \varepsilon \frac{\partial \vec{E}}{\partial t} \n\]

where \( \vec{E} \) is the electric field, \( \vec{B} \) is the magnetic field, \( \rho \) is the charge density, \( \varepsilon \) is the permittivity, and \( \mu \) is the permeability of the medium.

A function with a large curl

\( \vec{f}(x, y, z) = (-y, x, 0) \)

\( \vec{f}(1, 0, 0) = (0, 1, 0) \)
\( \vec{f}(0, 1, 0) = (-1, 0, 0) \)
\( \vec{f}(-1, 0, 0) = (0, -1, 0) \)
\( \vec{f}(0, -1, 0) = (1, 0, 0) \)

\( \nabla \times \vec{f} = \left( \frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}, \frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x}, \frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y} \right) \)

\( = (0 - 0, 0 - 0, 1 - (-1)) \)
\( = (0, 0, 2) \)

So this function has a curl of \( 2 \hat{z} \)

Derivation of the Wave Equation from Maxwell’s Equations

Take \( \nabla \times \) of:

\( \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \)

\( \nabla \times [\nabla \times \vec{E}] = \nabla \times \left[ -\frac{\partial \vec{B}}{\partial t} \right] \)

Change the order of differentiation on the RHS:

\( \nabla \times [\nabla \times \vec{E}] = -\frac{\partial}{\partial t} \nabla \times \vec{B} \)
Derivation of the Wave Equation from Maxwell’s Equations (cont’d)

But: \[ \nabla \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \]

Substituting for \( \nabla \times \vec{B} \), we have:

\[ \nabla \times [\nabla \times \vec{E}] = - \frac{\partial}{\partial t} [\nabla \times \vec{B}] = - \frac{\partial}{\partial t} [\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}] \]

Or:

\[ \nabla \times [\nabla \times \vec{E}] = - \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \]

assuming that \( \mu \) and \( \epsilon \) are constant in time.

Lemma: \[ \nabla \times [\nabla \times \vec{f}] = \nabla (\nabla \cdot \vec{f}) - \nabla^2 \vec{f} \]

Proof: Look first at the LHS of the above formula:

\[ \nabla \times [\nabla \times \vec{f}] = \nabla \times \left( \frac{\partial f_x}{\partial y} - \frac{\partial f_y}{\partial x} \right) - \nabla^2 \vec{f} \]

Taking the 2nd \( \nabla \times \) yields:

x-component: \[ = \left( \frac{\partial^2 f_x}{\partial x^2} - \frac{\partial^2 f_y}{\partial y^2} \right) - \frac{\partial^2 f_z}{\partial x^2} \]

y-component: \[ = \left( \frac{\partial^2 f_x}{\partial x \partial y} - \frac{\partial^2 f_y}{\partial y \partial x} \right) - \frac{\partial^2 f_z}{\partial y \partial z} \]

z-component: \[ = \left( \frac{\partial^2 f_x}{\partial x \partial z} - \frac{\partial^2 f_y}{\partial y \partial z} \right) - \frac{\partial^2 f_z}{\partial z^2} \]

Derivation of the Wave Equation from Maxwell’s Equations (cont’d)

Using the lemma, \( \nabla \times [\nabla \times \vec{E}] = - \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \)

becomes: \[ \nabla (\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = - \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \]

If we now assume zero charge density, \( \rho = 0 \), then

\[ \nabla \cdot \vec{E} = 0 \]

and we’re left with the Wave Equation!

\[ \nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \]
Why light waves are transverse

Suppose a wave propagates in the x-direction. Then it’s a function of x and t (and not y or z), so all y- and z-derivatives are zero:

\[
\frac{\partial E_y}{\partial y} = \frac{\partial E_z}{\partial z} = \frac{\partial B_z}{\partial y} = \frac{\partial B_z}{\partial z} = 0
\]

Now, in a charge-free medium, \( \vec{V} \cdot \vec{E} = 0 \) and \( \vec{V} \cdot \vec{B} = 0 \)

that is,

\[
\frac{\partial E_y}{\partial x} + \frac{\partial E_x}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \quad \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0
\]

Substituting the zero values, we have:

\[
\frac{\partial E_y}{\partial x} = 0 \quad \text{and} \quad \frac{\partial B_x}{\partial x} = 0
\]

So the longitudinal fields are at most \textit{constant}, and not waves.

The magnetic-field strength in a light wave

Suppose a wave propagates in the x-direction and has its electric field in the y-direction. What is the strength of the magnetic field along the y-direction [so \( E_y = E_{y0} \) and \( E_z = E_{z0} \)]?

What is the direction of the magnetic field?

Use:

\[
-\frac{\partial B_y}{\partial t} = \vec{\nabla} \times \vec{E} = \begin{pmatrix} \frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} \\ \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} \\ \frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y} \end{pmatrix}
\]

So:

\[
-\frac{\partial B_y}{\partial t} = \begin{pmatrix} 0 \\ 0 \\ \frac{\partial E_x}{\partial x} \end{pmatrix}
\]

In other words:

\[
-\frac{\partial B_y}{\partial t} = \frac{\partial E_x}{\partial x}
\]

And the magnetic field points in the z-direction.

The magnetic-field direction in a light wave

Suppose a wave propagates in the x-direction and has its electric field in the y-direction. What is the strength of the magnetic field?

Start with:

\[
-\frac{\partial B_y}{\partial t} = \frac{\partial E_x}{\partial x} \quad \text{and} \quad E_y(x,t) = E_{y0} \exp \left[ i(kx - \omega t) \right]
\]

We can integrate:

\[
B_y(x,t) = B_y(x,0) - \int_0^t \frac{\partial E_x}{\partial x} \, dt
\]

Take \( B_y(x,0) = 0 \)

So:

\[
B_y(x,t) = -\frac{ik}{i\omega} E_{y0} \exp \left[ i(kx - \omega t) \right]
\]

But \( \omega / k = c \):

\[
B_y(x,t) = \frac{1}{c} E_y(x,t)
\]

An Electromagnetic Wave

The electric and magnetic fields are \textit{in phase}. snapshot of the wave at one time

The electric field, the magnetic field, and the k-vector are all perpendicular:

\[
\vec{E} \times \vec{B} \propto \vec{k}
\]
The Energy Density of a Light Wave

The energy density of an electric field is: \( U_E = \frac{1}{2} \varepsilon E^2 \)
The energy density of a magnetic field is: \( U_B = \frac{1}{2} \mu B^2 \)
Using \( B = E/c \), and \( c = \frac{1}{\sqrt{\varepsilon\mu}} \), which together imply that \( B = E\sqrt{\varepsilon\mu} \)
we have:
\[
\begin{align*}
U_B &= \frac{1}{2} \mu (E^2\varepsilon\mu) = \frac{1}{2} \varepsilon E^2 = U_E
\end{align*}
\]
Total energy density: \( U = U_E + U_B = \varepsilon E^2 \)
So the electrical and magnetic energy densities in light are equal.

Why we neglect the magnetic field

The force on a charge, \( q \), is:
\[
\vec{F} = \vec{F}_{\text{electrical}} + \vec{F}_{\text{magnetic}}
\]
\[
\vec{F} = q \vec{E} + q \vec{v} \times \vec{B}
\]
where \( \vec{v} \) is the charge velocity
Taking the ratio of the magnitudes of the two forces:
\[
\frac{F_{\text{magnetic}}}{F_{\text{electrical}}} = \frac{q\sqrt{\mu}}{qE} \leq \frac{\vec{v} \times \vec{B}}{\vec{v}E} = \frac{\nu B}{\nu B} \leq 1
\]
Since \( B = E/c \):
\[
\frac{F_{\text{magnetic}}}{F_{\text{electrical}}} \leq \frac{\nu}{c}
\]
So as long as a charge’s velocity is much less than the speed of light, we can neglect the light’s magnetic force compared to its electric force.

The Poynting Vector: \( \vec{S} = c^2 \varepsilon \vec{E} \times \vec{B} \)

The power per unit area in a beam.
Justification (but not a proof):
Energy passing through area \( A \) in time \( \Delta t \):
\[
\begin{align*}
\Delta U &= U \Delta V = U A \Delta t
\end{align*}
\]
So the energy per unit time per unit area:
\[
\begin{align*}
\frac{\Delta U}{\Delta t} &= \frac{U A \Delta t}{\Delta t} = U c \varepsilon E^2
\end{align*}
\]
\[
\begin{align*}
\Rightarrow \bar{E} \times \vec{B} \sim k
\end{align*}
\]
And the direction \( \vec{E} \times \vec{B} \sim k \) is reasonable.

The Irradiance (often called the Intensity)

A light wave’s average power per unit area is the irradiance.
\[
\langle \vec{S}(\vec{r},t) \rangle = \frac{1}{T} \int_{T/2}^{+T/2} \vec{S}(\vec{r},t') \, dt'
\]
Substituting a light wave into the expression for the Poynting vector, \( \vec{S} = c^2 \varepsilon \vec{E} \times \vec{B} \), yields:
\[
\vec{S}(\vec{r},t) = c^2 \varepsilon [\vec{E}_0 \times \vec{B}_0] \cos^2(k \cdot \vec{r} - \omega t - \theta)
\]
The average of \( \cos^2 \) is 1/2:
\[
\Rightarrow I(\vec{r},t) = \left| \vec{S}(\vec{r},t) \right| = c^2 \varepsilon \left| \vec{E}_0 \times \vec{B}_0 \right|(1/2)
The Irradiance (continued)

Since the electric and magnetic fields are perpendicular and \( B_y = E_0 / c \),
\[
I = \frac{1}{2} c^2 E \left| \vec{E}_0 \times \vec{B}_0 \right| \quad \text{becomes:} \quad I = \frac{1}{2} c E \left| \vec{E}_0 \right|
\]

or:
\[
I = \frac{1}{2} c E \left| \vec{E}_0 \right|^2
\]

because the real amplitude squared is the same as the magnitude-squared complex one.

where:
\[
\left| \vec{E}_0 \right|^2 = E_{0x} E_{0x}^* + E_{0y} E_{0y}^* + E_{0z} E_{0z}^*
\]

Remember: this formula only works when the wave is of the form:
\[
\vec{E}(\vec{r}, t) = \text{Re} \left( \vec{E}_0 \exp \left[ i \left( \vec{k} \cdot \vec{r} - \omega t \right) \right] \right)
\]

that is, when all the fields involved have the same \( \vec{k} \cdot \vec{r} - \omega t \)

Sums of fields: Electromagnetism is linear, so the principle of Superposition holds.

If \( E_1(x, t) \) and \( E_2(x, t) \) are solutions to the wave equation, then \( E_1(x, t) + E_2(x, t) \) is also a solution.

Proof:
\[
\frac{\partial^2 (E_1 + E_2)}{\partial x^2} = \frac{\partial^2 E_1}{\partial x^2} + \frac{\partial^2 E_2}{\partial x^2} \quad \text{and} \quad \frac{\partial^2 (E_1 + E_2)}{\partial t^2} = \frac{\partial^2 E_1}{\partial t^2} + \frac{\partial^2 E_2}{\partial t^2}
\]

\[
\frac{\partial^2 (E_1 + E_2)}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 (E_1 + E_2)}{\partial t^2} = \frac{\partial^2 E_1}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 E_1}{\partial t^2} + \frac{\partial^2 E_2}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 E_2}{\partial t^2} = 0
\]

This means that light beams can pass through each other.

It also means that waves can constructively or destructively interfere.

Computing the Irradiance (Intensity)

Sometimes, you’re given the electric field, in which case the previous result works.

But sometimes, you’re given the energy \( (U) \), duration \( (\Delta t) \), (or power, \( P = U/\Delta t \)), and area \( (A) \). Then it’s easy:
\[
I = \frac{U}{(A \Delta t)}
\]

or:
\[
I = \frac{P}{A}
\]

The irradiance of the sum of two waves

If they’re both proportional to \( \exp \left[ i (\vec{k} \cdot \vec{r} - \omega t) \right] \), then the irradiance is:
\[
I = \frac{1}{2} c E \vec{E}_0 \cdot \vec{E}_0^* = \frac{1}{2} c E \left[ E_{ox} E_{ox}^* + E_{oy} E_{oy}^* + E_{oz} E_{oz}^* \right]
\]

Different polarizations (say \( x \) and \( y \)): \( \text{Intensities add.} \)
\[
I = \frac{1}{2} c E \left[ E_{ox} E_{ox}^* + E_{oy} E_{oy}^* \right] = I_x + I_y
\]

Same polarizations (say \( E_{ox} = E_1 + E_2 \)):
\[
I = \frac{1}{2} c E \left[ E_1 E_1^* + 2 \text{Re} \left( E_1 E_2^* \right) + E_2 E_2^* \right]
\]

Therefore:
\[
I = I_1 + cE \text{Re} \left( E_1 E_2^* \right) + I_2 \quad \text{Note the cross term!}
\]

The cross term is the origin of interference!
Interference only occurs for beams with the same polarization.
The irradiance of the sum of two waves of different color

Waves of different color (frequency) do not interfere!

We can't use the formula because the \( k \)'s and \( \omega \)'s are different.
So we need to go back to the Poynting vector, \( \hat{S}(\vec{r}, t) = \{ c^2 \mathbf{E} \times \mathbf{B} \} \)

\[
\hat{S}(\vec{r}, t) = c^2 \mathbf{E} \left[ \mathbf{E}_1 \mathbf{E}_2 \mathbf{E}_3 \mathbf{E}_4 \mathbf{E}_5 \mathbf{E}_6 \mathbf{E}_7 \mathbf{E}_8 \mathbf{E}_9 \mathbf{E}_{10} \right] \times \mathbf{B} + \mathbf{B} \]

This product averages to zero, as does \( \mathbf{E}_2 \times \mathbf{B} \)

Different colors: \( I = I_1 + I_2 \) Intensities add.

Irradiance of a sum of two waves

<table>
<thead>
<tr>
<th>Same polarizations</th>
<th>Different polarizations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Same colors</td>
<td></td>
</tr>
<tr>
<td>( I = I_1 + I_2 )</td>
<td>( I = I_1 + I_2 )</td>
</tr>
<tr>
<td>Different colors</td>
<td></td>
</tr>
<tr>
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</tbody>
</table>

Interference only occurs when the waves have the same color and polarization.